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A Direct Method to Evaluate Stress Intensity Factors Using Extended BEM

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ABSTRACT

The importance of stress intensity factors (SIFs) in linear elastic fracture mechanics is that they may be readily applied to give fracture safety assessment as well as fatigue life predictions. There are many numerical approaches that aim to capture the stress singularity in some way, and some postprocessing is required to give the SIFs. The proposed method provides a direct evaluation of SIFs with high accuracy and a low number of elements. The solution column reveals the values of K_I and K_{II} without the necessity for postprocessing calculations such as the J-integral. The method takes advantage of a new Extended BEM approach, which limits the additional Degrees of Freedom (DOF) to two per crack-tip; this allows for unlimited elements to be enriched. Auxiliary equations are derived from enforcing continuity of displacement at the crack tip. Numerical examples for mode I and mixed mode problems show a high level of accuracy with a low number of elements.

Key Words: *Enriched BEM, XBEM; Fracture Mechanics, Stress Intensity Factors*

1. Introduction

Cracks existing in engineering structures can grow extremely rapidly when they reach a certain length, leading to serious failure. Determination of crack growth rates and critical length requires an accurate evaluation of the stresses near to the crack-tip. In linear elastic fracture mechanics, William's expansions can provide an accurate evaluation of stresses near the crack-tip [1] once the SIFs are known. However, although analytical SIFs are available for simple geometries, for the majority of cases numerical techniques are required. Contributions of numerical methods in fracture mechanics are well-known. However, some selected previous works that have led to our algorithm for direct calculation of SIFs are briefly discussed.

The Partition of Unity Method (PUM) established the concept of using basis functions with better approximation properties than piecewise polynomials [2]; this has been widely implemented with FEM to model fracture mechanics problems with great success. The use of Extended FEM (XFEM) [3] has led to accurate results being produced from a coarse mesh. Simpson [4] has introduced a technique that might be called the extended BEM (XBEM), to determine SIFs with similar enrichment to XFEM. This shows a high accuracy at low computational rate, but requires a J-integral for accurate SIFs. However, the current method involves revealing accurate values of K_I and K_{II} directly, without the need for such postprocessing. This is likely to be of significance particularly in 3D, though this paper considers only 2D. A new auxiliary equation is also introduced to enforce displacement continuity at the crack tip, replacing the need for additional collocation points as used in [4]. The method shows flexibility in the number of enriched elements, which allows more elements to be enriched without increasing the DoFs or degrading the conditioning.

2. Formulation

The formulation of XBEM introduced by Simpson and Trevelyan [**?**], based on PUM, is used in the same fashion in the current method. The XBEM permits the use of Williams' asymptotic displacement expansions around the crack-tip as an additional basis function. Here the Williams displacement equation can be written as,

$$u_j = K_I \psi_{Ij}(\rho, \theta) + K_{II} \psi_{IIj}(\rho, \theta) \tag{1}$$

where K_I and K_{II} are mode I and mode II SIFs; $\psi_{Ij}(\rho, \theta)$ and $\psi_{IIj}(\rho, \theta)$ are given by the following functions, obtained from Williams expansions,

$$\psi_{Ix} = \frac{1}{2\mu}\sqrt{\frac{\rho}{2\pi}}\cos\frac{\theta}{2}\left[\kappa - 1 + 2\sin^2\frac{\theta}{2}\right] \tag{2a}$$

$$\psi_{IIx} = \frac{1}{2\mu}\sqrt{\frac{\rho}{2\pi}}\sin\frac{\theta}{2}\left[\kappa + 1 + 2\cos^2\frac{\theta}{2}\right] \tag{2b}$$

$$\psi_{Iy} = \frac{1}{2\mu}\sqrt{\frac{\rho}{2\pi}}\sin\frac{\theta}{2}\left[\kappa + 1 - 2\cos^2\frac{\theta}{2}\right] \tag{2c}$$

$$\psi_{IIy} = \frac{1}{2\mu}\sqrt{\frac{\rho}{2\pi}}\cos\frac{\theta}{2}\left[\kappa - 1 - 2\sin^2\frac{\theta}{2}\right] \tag{2d}$$

where ρ and θ are polar coordinates centred at the crack-tip, and κ is a parameter defined as $\kappa = 3 - 4\upsilon$ and $\kappa = \frac{3-\upsilon}{1+\upsilon}$ for plane strain and plane stress, respectively, υ being Poisson's ratio. Eq (1) can be rewritten to approximate the displacement near the crack-tip in the style of Benzley [5]; as follows,

$$u_j = \tilde{K}_I \psi_{Ij} + \tilde{K}_{II} \psi_{IIj} + \sum_{a=1}^{M} N^a u_j^a \tag{3}$$

where u_j^a is the nodal displacement, N^a is the Lagrangian shape function for node a and M is the total number of element nodes. $\tilde{K_I}$ and $\tilde{K_{II}}$ are coefficients playing the role of SIFs K_I and K_{II}, and are yielded as part of the solution vector. The first two terms of Eq. (3) are used to capture the local crack displacement, relative to the crack-tip, while the last term is included to approximate any non-zero displacement of the crack-tip.

The enriched displacement form (3) is used within the Dual Boundary Element Method (DBEM)[6]. DBEM is an ideal technique to model crack problems without giving rank deficiency. The BEM system is formed using the Displacement Boundary Integral Equation (DBIE) when collocating on one crack surface (and all non-crack boundaries) and the Traction Boundary Integral Equation (TBIE) for the opposing crack surface. DBIE can be written in a discretised form as,

$$C_{ij}(\grave{x})u_j(\grave{x}) + C_{ij}(\hat{x})u_j(\hat{x}) + \sum_{n=1}^{Ne}\sum_{a=1}^{M} P_{ij}^{na} u_j^{na} + \sum_{a=1}^{Ne}\sum_{l=1}^{2} \tilde{P}_{ijl}^{n} \tilde{K}_l = \sum_{n=1}^{Ne}\sum_{a=1}^{M} Q_{ij}^{na} t_j^{na} \tag{4}$$

where

$$P_{ij}^{na} = \int_{-1}^{1} N^a(\xi) T_{ij}(\grave{x}, x(\xi)) J^n(\xi) d\xi \tag{5a}$$

$$Q_{ij}^{na} = \int_{-1}^{1} N^a(\xi) U_{ij}(\grave{x}, x(\xi)) J^n(\xi) d\xi \tag{5b}$$

$$\tilde{P}_{ijl}^{n} = \int_{-1}^{1} T_{ij}(\grave{x}, x(\xi)) \psi_{lj}(\xi) J(\xi) d\xi \qquad \text{where } l = I \text{ and } II \tag{5c}$$

where Ne is the total number of elements, and $J^n(\xi)$ is the Jacobian. T_{ij}, U_{ij} are the usual traction and displacement kernels. If the n^{th} element is unenriched then $\tilde{P}_{ijl}^{n} = 0, l = I, II$. In addition, as $\theta = \pm\pi$ at the crack surfaces for flat cracks, ψ_{Ij} and ψ_{IIj} are only functions of ξ . Jump terms in the enriched DBIE remain the same as enriched jump terms; these will cancelled during implementation. The TBIE can be obtained in numerical form as follows,

$$n_i(\grave{x}) \sum_{n=1}^{Ne}\sum_{a=1}^{M} E_{kij}^{na} u_k^{na} + n_i(\grave{x}) \sum_{a=1}^{Ne}\sum_{l=1}^{2} \tilde{E}_{kijl}^{n} \tilde{K}_l = n_i(\grave{x}) \sum_{n=1}^{Ne}\sum_{a=1}^{M} F_{kij}^{na} t_k^{na} \tag{6}$$

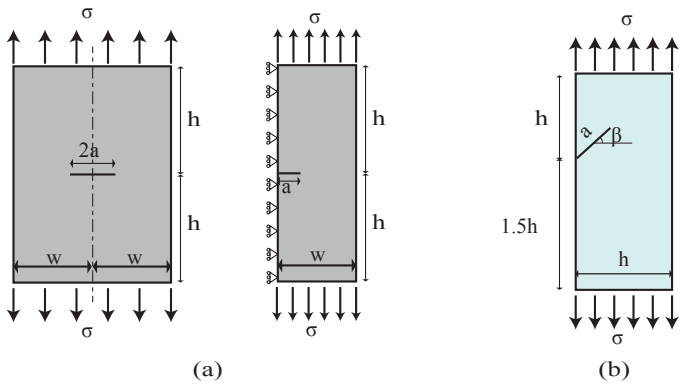

Figure 1: Numerical examples, *a*) Mode I centred crack on a flat plate; *b*) Inclined edge crack on rectangular plate

where

$$E^{na}_{kij} = \int_{-1}^{1} N^a(\xi) S_{kij}(\grave{x}, x(\xi)) J^n(\xi) d\xi \tag{7a}$$

$$F^{na}_{kij} = \int_{-1}^{1} N^a(\xi) D_{kij}(\grave{x}, x(\xi)) J^n(\xi) d\xi \tag{7b}$$

$$\tilde{E}^{n}_{kijI} = \int_{-1}^{1} S_{kij}(\grave{x}, x(\xi)) \psi_{lk}(\xi) J(\xi) d\xi \qquad \text{where } l = I \text{ and } II \tag{7c}$$

where S_{kij}, D_{kij} are the usual derivative kernels. If the n^{th} element is unenriched $\tilde{E}^{n}_{kijl} = 0, l = I, II$. Implementation of the TBIE and DBIE requires considerable care in evaluating the hyper-singular and strongly-singular integrals, and the injection of extra enrichment degrees of freedom requires us to supply auxiliary equations to reach a square linear system.

3. Crack-tip displacement continuity

The Dual BEM involves a hypersingular integral equation which places requirements on the continuity of displacement derivatives at the collocation point. This cannot normally be achieved because of the C^0 continuity of shape functions at nodes shared between adjacent elements. Therefore, most Dual BEM implementations make use of discontinuous elements. One result of this is that a displacement discontinuity is often observed at the crack tip. While this does not in itself preclude us from obtaining accurate SIFs, it does provide us with an opportunity to design a simple set of auxiliary equations while at the same time enforcing a displacement continuity that is observed in the physical problem being modelled.

The crack tip displacement can be approximated by extrapolating over the adjacent elements on the upper and lower crack surfaces. The approximations taken from the two surfaces can be equated to enforce displacement continuity, i.e.

$$\sum_{a=1}^{L} u^a_{j(upper)} N^a = \sum_{a=1}^{L} u^a_{j(lower)} N^a \tag{8}$$

L is the number of nodes used for the crack-tip extrapolation and N^a is the associated Lagrangian shape function. Eq. (8) is considered in both x and y directions independently, and the resulting equations used to form extra rows of the matrix description of the BEM problem. Now, $\tilde{K_I}$ and $\tilde{K_{II}}$ can present the SIFs directly, without requiring the J-integral.

4. Numerical Examples

The first example is a pure Mode I centre crack in a rectangular flat plate under uniaxial traction, with dimensions of $h = 2w = 4a$, as illustrated in Fig.1(a). Because of symmetry, only half of the plate is considered. The problem does not have an exact solution; instead a reference solution is used[7]. Normalised results K/K_0, where $K_0 = \sigma\sqrt{\pi a}$, are shown in Fig. 2 in which the reference solution is plotted as a horizontal line.

In the second example, a mixed mode inclined edge crack in a rectangular flat plate is shown in Fig. 1(b). A solution by Wilson [8] is used as a reference solution. Results have been plotted in Figs. 3 (a) and (b) for K_I and K_{II}, respectively.

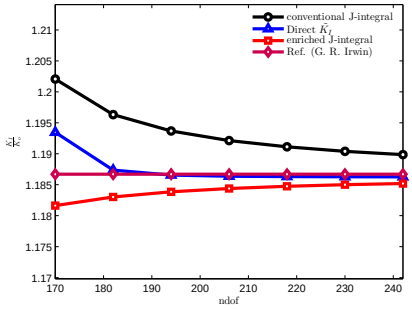

Figure 2: K_I compared to post-process method

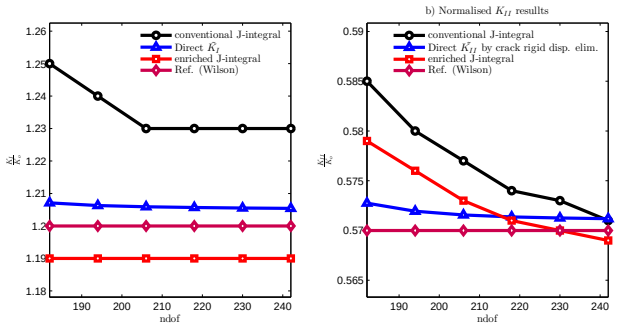

Figure 3: K_I compared to post-process method

In all figures we compare the Direct $\tilde{K}_I$ and $\tilde{K}_{II}$ results against (i) a conventional (unenriched) Dual BEM using the J-integral, and (ii) an enriched Dual XBEM using the J-integral. In all cases the directly obtained SIFs show excellent convergence properties.

5. Conclusions

A direct method has been presented that is able to evaluate SIFs without need for postprocessing (J-integral) calculations. This has been achieved using a Dual XBEM approach in which auxiliary equations are formed by enforcing displacement continuity at the crack tip. Additional DoFs are limited to two per crack-tip, which allows for the enrichment of all crack surface elements for greater accuracy with controlled conditioning. Special treatment must be applied to singular integrals as illustrated in [4]. The results show accurate values for SIFs compared to conventional J-integral based BEM approaches. The method can be extended to 3D, where the removal of the requirement for a J-integral will be helpful.

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