A TWO MODE NON-UNIFORM APPROXIMATION FOR AN ELASTIC ASYMMETRIC SANDWICH

Mohammed Alkinidri, Julius Kaplunov, and Ludmila Prikazchikova

Keele University
Staffordshire, UK, ST5 5BG
e-mail: m.o.s.alkinidri@keele.ac.uk, j.kaplunov@keele.ac.uk, l.prikazchikova@keele.ac.uk

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Abstract. An elastic asymmetric sandwich is considered under the assumption that the stiffness and density of thin skin layers are much greater than those of a core layer. It is shown that for typical values of problem parameters the lowest shear resonance frequency appears to be asymptotically small, while the rest of thickness resonances do not belong to a low-frequency range. As an example, a scalar antiplane problem in linear elasticity is studied. A polynomial long-wave low-frequency approximation of the full dispersion relation is derived. It governs two vibration modes including the fundamental one and the lowest harmonic. Without asymmetry the dispersion relation splits into two parts corresponding to symmetric and antisymmetric modes [1], simplifying analysis drastically. It is remarkable that for the chosen set of problem parameters the derived approximation is not asymptotically uniform and is only valid over narrow non-overlapping vicinities of zero and the lowest shear thickness resonance frequencies. This is in contrast to the earlier considered setup of a similar laminate with the thick skin layers, for which the associated asymptotic behaviour is uniform [2]. The same observation is also true for in-plane motion of a symmetric laminate, see [3] for further detail.
1 Introduction

Asymmetric sandwich plates become increasingly important as components of ultra-light multifunctional structures in various fields of engineering, see for example [4, 5, 6] and references therein. Many of these structures might also demonstrate high contrast in material and geometric properties of the layers, for example lightweight photovoltaic panels [7] and new models of windscreens [8].

Asymptotic analysis of an asymmetric sandwich structure with a light core and stiff skin layers is conducted in [2] for anti-plane shear motion. As shown in this paper, an additional small cut-off frequency might arise depending on the choice of the contrast material parameters of the layers. A two-mode long-wave asymptotic approximation, constructed in [2] is uniform, i.e. valid for the whole low-frequency range. At the same time, despite the presence of a small cut-off frequency, it might not always be the case. There are some setups of contrasting problem parameters resulting in non-uniform asymptotic behaviour, e.g. see [3] considering a plane problem for a symmetric sandwich plate. In this case, the associated asymptotic behaviour appears to be composite, i.e. it is valid only over narrow non-intersecting vicinities of zero and lowest cut-off frequencies. The concept of composite expansions is based on tackling limiting cases only without a special care of the less important intermediate region, e.g. see [10] and [11]. Recently, composite wave models have been constructed for thin and periodic structures in [12, 13, 14]. An asymptotic methodology for three-layered plates with contrast parameters of the layers was earlier adapted in [15] mainly concerned with static deformation. Among the publications on the subject, we also mention [16, 17, 18, 19, 20].

In this paper we investigate the anti-plane problem for a three-layered asymmetric plate with very light core layer and stiff thin skin layers of different thickness. A two-mode long-wave low-frequency approximation of the exact dispersion relation is constructed and shown to be non-uniform. The asymptotic results are compared with the exact solution.

2 Statement of the problem

Consider a three-layered asymmetric elastic laminate, comprised of isotropic layers of thickness $h_l$, $l = 1, 2, 3$, see Figure 1. The axis $x_1$ of the Cartesian coordinate system passes through the mid-plane of the core layer, while the axis $x_2$ is orthogonal to the mid-plane. For the sake of simplicity we assume that the two outer layers have the same material parameters.

Consider antiplane shear motion, for which the only non-zero displacement component is orthogonal to the plane $x_1x_2$. The equations of motion for each of the layers is then given by

$$
\Delta u_l - \frac{1}{c_{2l}^2} \frac{\partial^2 u_l}{\partial t^2} = 0, \quad l = 1, 2, 3, \tag{1}
$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator, $u_l = u_l(x_1, x_2, t)$ are out of plane displacements, $t$ is time, and

$$
c_{2l} = \sqrt{\frac{\mu_l}{\rho_l}},
$$

are the shear wave speeds, with $\mu_l$ and $\rho_l$ standing for the Lamé elastic moduli and volume mass densities, respectively. As mentioned above, $\mu_1 = \mu_3$ and $\rho_1 = \rho_3$. 

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For traction-free faces and perfect bonding between the layers, the dispersion relation may be derived in the form
\[
\mu \alpha_1 \alpha_2 \left( \tanh(h_{12} \alpha_1) + \tanh(h_{32} \alpha_1) \right) + \mu^2 \alpha_2^2 \tanh(\alpha_2) + \\
+ \alpha_1^2 \tanh(h_{12} \alpha_1) \tanh(h_{32} \alpha_1) \tanh(\alpha_2) = 0
\] (2)
where
\[
\alpha_1 = \sqrt{K^2 - \frac{\mu}{\rho} \Omega^2}, \quad \alpha_2 = \sqrt{K^2 - \Omega^2}
\]
and
\[
K = kh_2, \quad \Omega = \frac{\omega h_2}{c_{22}}, \quad \mu = \frac{\mu_2}{\mu_1}, \quad \rho = \frac{\rho_2}{\rho_1}, \quad h_{12} = \frac{h_1}{h_2}, \quad h_{32} = \frac{h_3}{h_2}.
\]

3 Asymptotic approach

Consider an asymmetric laminate of a sandwich type, with the outer layers being stiff, thin and very heavy compared to the inner core layer. Hence, the ratio of parameters in the layers are taken as
\[
\mu \ll 1, \quad \rho \sim \mu^2, \quad h_{12} \sim \mu, \quad h_{32} \sim \mu.
\] (3)

Setting $K = 0$ in (2) we obtain as expected $\Omega = 0$, and also a small non-zero cut-off frequency given by
\[
\Omega_{sh} \approx \sqrt{\frac{(h_{12} + h_{32})\rho}{h_{12}h_{32}}} \sim \sqrt{\mu} \ll 1.
\] (4)

Next, expanding the trigonometric functions in (2) in Taylor series about $\Omega = K = 0$ and assuming (3), an approximate polynomial dispersion relation may be obtained in the form
\[
\gamma_1 K^2 + \gamma_2 \Omega^2 + \gamma_3 K^4 + \gamma_4 K^2 \Omega^2 + \gamma_5 \Omega^4 + \gamma_6 K^6 \\
+ \gamma_7 K^4 \Omega^2 + \gamma_8 K^2 \Omega^4 + \gamma_9 \Omega^6 + \cdots = 0
\] (5)
where the coefficients $\gamma_i$, $i = 1, 2, \ldots 9$ have been presented in [2] (cf. formula [14] in the cited paper).
Now, in view of (3), leading order coefficients $\gamma_i$ become

$$\gamma_1 = \gamma_1^0 \mu^2, \quad \gamma_1^0 = 1 + h_{12}^0 + h_{32}^0,$$

$$\gamma_2 = \gamma_2^0 \mu + O(\mu^2), \quad \gamma_2^0 = -\frac{h_{12}^0 + h_{32}^0}{\rho_0},$$

$$\gamma_3 = \gamma_3^0 \mu^2 + O(\mu^4), \quad \gamma_3^0 = h_{12}^0 h_{32}^0 - \frac{1}{3},$$

$$\gamma_4 = \gamma_4^0 \mu + O(\mu^2), \quad \gamma_4^0 = -\frac{2h_{12}^0 h_{32}^0}{\rho_0},$$

$$\gamma_5 = \gamma_5^0 + O(\mu^2), \quad \gamma_5^0 = \frac{h_{12}^0 h_{32}^0}{\rho_0^2},$$

$$\gamma_6 = \gamma_6^0 \mu^2 + O(\mu^4), \quad \gamma_6^0 = \frac{2}{15} - \frac{h_{12}^0 h_{32}^0}{3 \rho_0},$$

$$\gamma_7 = \gamma_7^0 \mu + O(\mu^2), \quad \gamma_7^0 = \frac{2h_{12}^0 h_{32}^0}{3 \rho_0},$$

$$\gamma_8 = \gamma_8^0 + O(\mu), \quad \gamma_8^0 = -h_{12}^0 h_{32}^0, \quad \gamma_8^0 = \frac{h_{12}^0 h_{32}^0}{3 \rho_0},$$

$$\gamma_9 = \gamma_9^0 + O(\mu), \quad \gamma_9^0 = \frac{h_{12}^0 h_{32}^0}{3 \rho_0^2},$$

where $\rho_0 = \rho / \mu^2$, $h_{12}^0 = h_{12} / \mu$ and $h_{32}^0 = h_{32} / \mu$. These results are summarised in Table 1, allowing comparison of asymptotic orders of the terms in approximation (5) both in the vicinity of zero and the lowest cut-off frequency (4).

<table>
<thead>
<tr>
<th>Order of $\gamma_i$</th>
<th>Terms</th>
<th>Fundamental mode $\Omega^2 \sim \mu K^2$</th>
<th>First harmonic $\Omega_{sh}^2 \sim \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1 \sim \mu^2$</td>
<td>$\gamma_1 K^2$</td>
<td>$\mu^2 K^2$</td>
<td>$\mu^2 K^2$</td>
</tr>
<tr>
<td>$\gamma_2 \sim \mu$</td>
<td>$\gamma_2 K^2$</td>
<td>$\mu^2 K^2$</td>
<td>$\mu^2 K^2$</td>
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<tr>
<td>$\gamma_3 \sim \mu^2$</td>
<td>$\gamma_3 K^4$</td>
<td>$\mu^2 K^4$</td>
<td>$\mu^2 K^4$</td>
</tr>
<tr>
<td>$\gamma_4 \sim \mu$</td>
<td>$\gamma_4 K^4 \Omega^2$</td>
<td>$\mu^2 K^4$</td>
<td>$\mu^2 K^4$</td>
</tr>
<tr>
<td>$\gamma_5 \sim 1$</td>
<td>$\gamma_5 \Omega^4$</td>
<td>$\mu^2 K^4$</td>
<td>$\mu^2 K^4$</td>
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<tr>
<td>$\gamma_6 \sim \mu^2$</td>
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<tr>
<td>$\gamma_7 \sim \mu$</td>
<td>$\gamma_7 K^4 \Omega^2$</td>
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<tr>
<td>$\gamma_8 \sim 1$</td>
<td>$\gamma_8 K^4 \Omega^4$</td>
<td>$\mu^2 K^6$</td>
<td>$\mu^2 K^6$</td>
</tr>
<tr>
<td>$\gamma_9 \sim 1$</td>
<td>$\gamma_9 \Omega^6$</td>
<td>$\mu^2 K^6$</td>
<td>$\mu^2 K^6$</td>
</tr>
</tbody>
</table>

Table 1: Asymptotic behaviour at $\mu \ll 1$, $\rho \sim \mu^2$, $h_{12} \sim h_{32} \sim \mu$

Using this Table, the leading order shortened approximate dispersion relation may be constructed, incorporating the fundamental mode along with the lowest harmonic with the asymptotically small cut-off frequency (4). It may be expressed as

$$\mu^2 \gamma_1^0 K^2 + (\mu \gamma_2^0 + \gamma_5^0 \Omega^2) \Omega^2 + \mu \gamma_4^0 K^2 \Omega^2 + \gamma_8^0 K^2 \Omega^4 + \gamma_9^0 \Omega^6 = 0. \quad (7)$$
In this formula all the terms are of the same order $\mu^3$ at $\Omega - \Omega_{sh} \sim \mu^{3/2}$, $K \sim \sqrt{\mu}$. The local asymptotic approximation for the fundamental mode is given by

$$\mu \gamma_1^0 K^2 + \gamma_2^0 \Omega^2 = 0.$$  \hfill (8)

At the same time, the local expansion for the first harmonic becomes

$$\mu^2 \gamma_1^0 K^2 + \left( \mu \gamma_1^0 + \gamma_5^0 \Omega^2 \right) \Omega_{sh}^2 + \mu \gamma_4^0 K^2 \Omega_{sh}^2 + \gamma_8^0 K^2 \Omega_{sh}^4 + \gamma_9^0 \Omega_{sh}^6 = 0,$$  \hfill (9)

where $\Omega_{sh}$ is given by (4). The associated typical near cut-off expansion, e.g. see [21, 22] for further details, is

$$\Omega^2 - \Omega_{sh}^2 = -K^2 \frac{1}{\gamma_5^0} \left( \mu^2 \gamma_1^0 \Omega_{sh}^2 + \mu \gamma_4^0 + \gamma_8^0 \Omega_{sh}^2 \right) - \frac{\gamma_9^0 \Omega_{sh}^4}{\gamma_5^0}.$$  \hfill (10)

Note that the long-wave assumption $K \ll 1$ dictates that the approximation (8) for the fundamental mode is valid only for $\Omega \ll \sqrt{\mu}$, whereas (10) is associated with $\Omega \sim \sqrt{\mu}$. Thus, the approximation (7) is non-uniform, in line with previous considerations in [3].

Numerical illustrations of the derived approximations are given in Figures 2-3. In Figure 2 two-mode approximation (7) (dotted lines) is compared numerically with the exact solution of the dispersion relation (2) (solid lines). A characteristic gap where the asymptotic formula (7) is not applicable is shown in Figure 2.

![Figure 2: Dispersion curves (2) (solid lines) together with approximation (7) (dotted lines) for $h_{12} = 0.1$, $h_{32} = 0.2$, $\mu = 0.1$, and $\rho = 0.01$.](image-url)

In Figure 3 exact dispersion curves (2) (solid lines) are shown together with approximations for the fundamental mode (8) and for the lowest harmonic (10) (dotted lines), demonstrating high accuracy of asymptotic predictions.
Figure 3: Dispersion curves (2) (solid lines) together with approximations for a) the fundamental mode (8) and b) the lowest harmonic (10) (dotted lines) for $h_{12} = 0.1$, $h_{32} = 0.2$, $\mu = 0.1$ and $\rho = 0.01$

4 Conclusion

The present analysis complements the previous results for an asymmetric layered plate in [2]. Two-mode approximate dispersion relation has been derived for a chosen scenario for which the outer layers are relatively stiff, thin and very heavy compared to the inner layer. In contrast to [2], in which the two-mode long-wave low-frequency approximation was asymptotically uniform, the approximation obtained in the paper is of composite nature, being only valid over non-overlapping vicinities of the origin and the first cut-off frequency. Thus, the effect of high contrast does not always result in a uniform two-mode approximation as might be expected. The proposed research may be expanded to the derivation of the related two-mode differential equations of motion, based on the same scaling.

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REFERENCES


